Eikonal Method for Calculation of Coherence Functions

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(Received 3 March 2005; published 22 July 2005)

A method is presented for computing the cross-spectral density of a special class of partially coherent fields in which the coherent modes obey an eikonal equation. This method allows for statistical analysis of optical systems based on simple ray tracing.

DOI: 10.1103/PhysRevLett.95.043904 PACS numbers: 42.15.−i, 42.25.Kb

The calculation of observable quantities in wave physics often requires a statistical approach. In optics the measurable quantities may always be related to correlation functions of the field. Likewise in quantum mechanics, practical calculations must often be made using a density matrix approach. It is well known that the second order correlation functions of statistically stationary wave fields are governed by the so-called Wolf equations. For scalar fields in the frequency domain, the Wolf equations take the form [1]

$$[\nabla_j^2 + k^2 n^2(\mathbf{r}_j)]W(\mathbf{r}_1, \mathbf{r}_2, \nu) = 0, \tag{1}$$

where $j = 1, 2$, W is the cross-spectral density, $k = 2\pi\nu/c$ is the free-space wave number, and $n(\mathbf{r})$ is the index of refraction. The cross-spectral density may be propagated from an arbitrary closed surface by the usual method of Green's functions applied over both coordinates:

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \frac{1}{(2\pi)^2}\int_S d^2 r_1' \int_S d^2 r_2' W(\mathbf{r}_1', \mathbf{r}_2', \nu) \times \frac{\partial}{\partial n_1'} G^*(\mathbf{r}_1, \mathbf{r}_1') \frac{\partial}{\partial n_2'} G(\mathbf{r}_2, \mathbf{r}_2'), \tag{2}$$

where S is a closed surface on which W is known, $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit vectors normal to S, and G is the Green's function for the wave equation satisfying appropriate boundary conditions. Exact evaluation of Eq. (2), except in the simplest of circumstances, is quite difficult. Simplifying approximations are often made; for example, analysis may be limited to the paraxial zone [2]. In special cases, these formulas have been evaluated by asymptotic methods [3].

In many applications, such as classical microscopy, the methods of geometrical optics (GO) provide a satisfactory means of computing the intensity for complicated optical systems. A number of commercially available software packages are used to compute transfer of intensity and optical path length in imaging and nonimaging optical systems. However, these methods fail to take into account the coherence properties of the field. Furthermore, in many modern techniques such as optical coherence tomography (OCT) [4], the propagated coherence functions are themselves of great importance. This raises an intriguing question. Can the methods of GO be applied in a more general setting to the propagation of coherence functions?

In this Letter, it is demonstrated that GO can be applied to the class of partially coherent fields whose coherent modes each obey an eikonal function. The coherent modes of any realizable field form a discrete set with decreasing weights so that a finite number of modes may be kept to adequately account for the propagation of fields in arbitrary optical systems. In the case that the field is completely coherent, one mode suffices and the problem reduces to the usual case of GO.

In contrast to the results presented here, a method was presented for calculation of the evolution of the spectral density of a partially coherent field [5]. It was claimed that the spectral density, here denoted $W(\mathbf{r}, \mathbf{r}, \nu)$, could be computed by the use of a single eikonal. It may be seen here that this claim is only true when the coherent mode decomposition is dominated by a single mode, i.e., the field is fully coherent.

The cross-spectral density of a partially coherent, statistically stationary field can be represented as a coherent mode decomposition [1]

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \sum_m \beta_m(\nu)\phi_m^*(\mathbf{r}_1, \nu)\phi_m(\mathbf{r}_2, \nu), \tag{3}$$

where $\beta_m(\nu)$ and $\phi_m(\mathbf{r}, \nu)$ are the eigenvalues and eigenfunctions, respectively, that satisfy the integral equation $\int d^3 r_1 W(\mathbf{r}_1, \mathbf{r}_2, \nu)\phi_m(\mathbf{r}_1, \nu) = \beta_m(\nu)\phi_m(\mathbf{r}_2, \nu)$. The cross-spectral density may be propagated via propagation of the individual modes. Let $\psi_m(\mathbf{r}, \nu)$ be the propagated version of $\phi_m(\mathbf{r}, \nu)$. Then, *due to the statistical independence of the modes* [1], the cross-spectral density may be written

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \sum_m \beta_m(\nu)\psi_m(\mathbf{r}_1, \nu)^*\psi_m(\mathbf{r}_2, \nu). \tag{4}$$

Consider a special class of coherent modes that take the form

$$\psi_m(\mathbf{r}, \nu) = U_m(\mathbf{r}) e^{ikS_m(\mathbf{r})}, \quad (5)$$

where $U_m(\mathbf{r})$ is a real quantity that is the square root of the spectral density associated with mode m, and $S_m(\mathbf{r})$ is the eikonal. Fields of this type may be treated *exactly* within the framework of GO [6,7]. In many cases of interest, the $\phi_m(\mathbf{r}, \nu)$ may be well approximated by Eq. (5), this form being obtained as the leading term in an asymptotic series for large values of k. The quantity $S_m(\mathbf{r})$ obeys the eikonal equation

$$|\nabla S_m(\mathbf{r})|^2 = n^2(\mathbf{r}). \quad (6)$$

Substituting the propagated modes from (5) into (4) yields

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \sum_m \beta_m(\nu) U_m(\mathbf{r}_1) U_m(\mathbf{r}_2) e^{ik\Delta S_m(\mathbf{r}_1, \mathbf{r}_2)}, \quad (7)$$

where $\Delta S_m(\mathbf{r}_1, \mathbf{r}_2) = S_m(\mathbf{r}_2) - S_m(\mathbf{r}_1)$. In the time domain, the cross-correlation function $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ is given by the Fourier transform of $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ with respect to ν. Γ is of particular importance in low-coherence imaging systems such as OCT. Thus Eq. (7) implies that

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, t) = \sum_m \tilde{\beta}_m\left(t - \frac{\Delta S_m(\mathbf{r}_1, \mathbf{r}_2)}{c}\right) U_m(\mathbf{r}_1) U_m(\mathbf{r}_2), \quad (8)$$

where c is the speed of light and $\tilde{\beta}_m(t)$ is the Fourier transform of $\beta_m(\nu)$ with respect to ν. Eqs. (7) and (8) are simple yet powerful results that show that the coherence functions can be found within the accuracy of GO. Thus the extensive theoretical and computational tools available for GO may be applied to problems in which the coherence properties of the field play a role.

The results above are illustrated by two examples. First, the propagation of light from three light-emitting diodes (LEDs) propagated through an optical system is considered. The LEDs are assumed to be incoherent point sources so that the coherent modes are the fields of point sources located at the position of the LEDs. Thus the standard methods for calculating point characteristics in geometrical optics may be employed. Without the optical system, the cross-spectral density is given by the expression

$$W(\mathbf{r}_1, \mathbf{r}_2) = \sum_{j=1}^{3} S_j(\nu) \frac{e^{ik(|\mathbf{r}_1 - \mathbf{r}_j| - |\mathbf{r}_2 - \mathbf{r}_j|}}{|\mathbf{r}_1 - \mathbf{r}_j||\mathbf{r}_2 - \mathbf{r}_j|},$$

where $S_j(\nu)$ is the spectrum of the jth LED. The optical system is taken to be a thin lens characterized by an *ABCD* matrix [8]. In this example, the point sources are located at $(-0.2, 0.0, -17)$ mm, $(0.0, 0.0, -8.0)$ mm, and $(0.6, 0.0, -10)$ mm in Cartesian coordinates centered at the midpoint of the lens. The LED colors are red, green, and yellow, respectively, and the spectra, shown in Fig. 1, have been taken from experiment. The lens has a diameter $d = 2$ mm and a focal length $f = 5$ mm. In Fig. 2 the cross-spectral density, the cross-correlation function, and the modulus of the spectral degree of coherence $\mu(\mathbf{r}_1, \mathbf{r}_2, \nu) = W(\mathbf{r}_1, \mathbf{r}_2, \nu)/\sqrt{W(\mathbf{r}_1, \mathbf{r}_1, \nu)W(\mathbf{r}_2, \mathbf{r}_2, \nu)}$ are shown at a fixed frequency ν in the plane located 6 mm from the lens. The contribution of each LED to the field at a given point is determined by the degree to which the field from that LED is focused as well as the spectrum of the LED. Note that abrupt changes arise from the presence of boundaries between shadow and lit regions of the LEDs. $\Gamma(\mathbf{r}, \mathbf{r}, t)$ is the temporal autocorrelation function of the field at point $\mathbf{r}$, which is of practical importance for low-coherence interferometric imaging [9]. These results imply that the autocorrelation function may be markedly different in different areas, thus potentially creating a nonuniform spatial response. Although it is expected that $\mu(\mathbf{r}_1, \mathbf{r}_2, \nu)$ will be equal to one on the line $\mathbf{r}_1 = \mathbf{r}_2$, this line is too narrow to appear in the plot in the shadow regions.

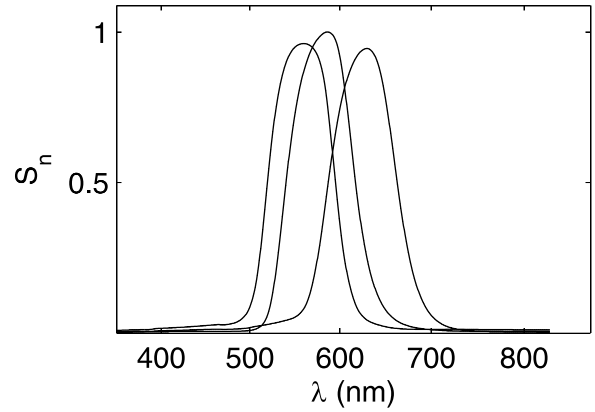

FIG. 1. The normalized spectral densities, S_n, for the three LEDs used in the first example.

As a second example, a partially coherent beam is propagated through a simple optical system and the coherence properties of the resultant field are considered. The field is described by a Gaussian Schell-model beam with a secondary source cross-spectral density

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = S_0(\nu) \exp\left[\frac{-(\mathbf{r}_1^2 + \mathbf{r}_2^2)}{4\sigma_s^2(\nu)} - \frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{2\sigma_g^2(\nu)}\right], \quad (9)$$

where $\sigma_s^2(\nu)$ and $\sigma_g^2(\nu)$ are the variances of the intensity and spectral degree of coherence [10]. The coherent modes are given by the expressions [11,12] $\phi_{lm}(x, y, \nu) = \phi_l(x, \nu)\phi_m(y, \nu)$ and $\beta_{lm}(\nu) = \beta_l(\nu)\beta_m(\nu)$, where

$$\phi_m(x, \nu) = e^{-c(\nu)x^2} H_m(x\sqrt{2c(\nu)}) \left(\frac{2c(\nu)}{\pi}\right)^{1/4} \left(\frac{1}{2^m m!}\right)^{1/2}, \quad (10)$$

and

$$\beta_m(\nu) = \left(\frac{b(\nu)}{a(\nu) + b(\nu) + c(\nu)}\right)^{m+1/2} \left(\frac{\pi S_0(\nu)}{b(\nu)}\right)^{1/2}. \quad (11)$$

Here, $H_m(x)$ is the Hermite polynomial of order m, $c(\nu) = [a(\nu)^2 + 2a(\nu)b(\nu)]^{1/2}$, $a(\nu) = 1/4\sigma_s^2(\nu)$, and $b(\nu) = 1/2\sigma_g^2(\nu)$. Note that the propagated modes are not of the

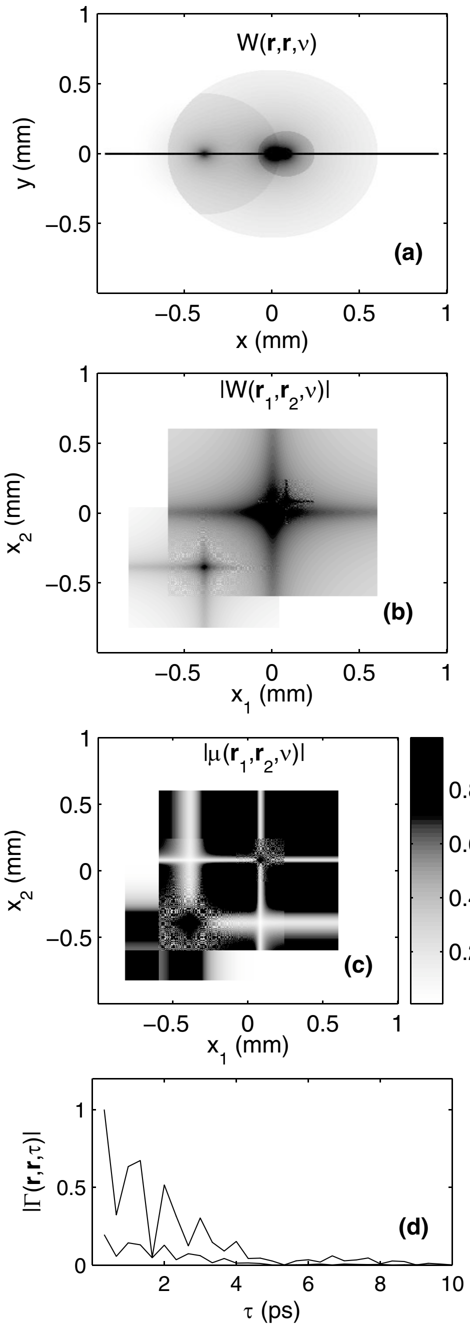

FIG. 2. (a) Log of the spectral density at $\nu = 5 \times 10^{14}$ Hz ($\lambda = 600$ nm) shown in arbitrary units. (b) Cross-spectral density at points along the line shown in part (a). (c) Modulus of the spectral degree of coherence at points along the line shown in part (a). (d) Cross-correlation as a function of time taken at $(x, y) = (0.3, 0.0)$ mm and $(x, y) = (-0.1, -0.1)$ mm.

form in (5), but satisfy a paraxial eikonal equation and may thus be treated approximately by GO. The effect of an optical system on this field may be computed via an *ABCD* matrix transformation applied to the Hermite-Gaussian modes in Eq. (10) [13]

$$\psi_m(x) = \left[\frac{2c(\nu)}{\pi}\right]^{1/4}\left(\frac{1}{2^m m!}\right)^{1/2}\Omega H_m(\Omega^2 x\sqrt{2c(\nu)}) \times e^{-ik(z/2+x^2/2\sigma^2)+i(m+1/2)\delta}, \tag{12}$$

where $\Omega = [A^2 + 2B^2c^2/k^2]^{-1/4}$, $\sigma^2 = (2ikcA + B)/(2ikcC + D)$, and $\delta = \tan^{-1}(2Bc/kA)$. For this example $\lambda = 632.8$ nm, $\sigma_g(\nu) = 0.5$ mm, $\sigma_s(\nu) = 2$ mm. The optical system is taken to be a thin lens ($f = 20$ mm) and the aperture is taken to be infinite so that the elements of the ray transfer matrix are given by $A = 1 - z/f$, $B = z$, $C = -1/f$, and $D = 1$. The cross-spectral density $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ and modulus of the spectral degree of coherence

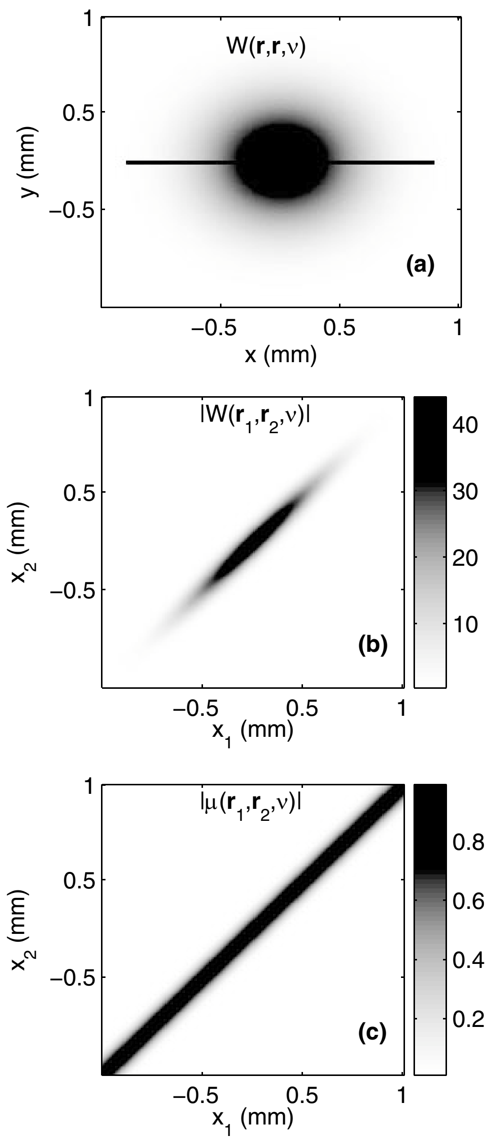

FIG. 3. (a) The spectral density normalized by the peak incident spectral density, $S_0(\nu)$. (b) Cross-spectral density similarly normalized, at points along the line shown in part (a). (c) Modulus of the spectral degree of coherence at points along the line shown in part (a). The plane of interest is located at $z = 7$ mm. $N = 60$ modes are displayed.

$\mu(\mathbf{r}_1, \mathbf{r}_2, \nu)$ of the resulting system ($z = 17$ mm, $y = 0$ mm) are shown in Fig. 3. It may be observed that the width of the spectral degree of coherence of the focused field is approximately 6 times smaller than that of the original beam as expected [14]. Beyond the GO approximation, the cross-spectral density may be computed by Eq. (2) [15], or if only the spectral density is of interest some simplification may be achieved [3,16,17].

In summary, it has been shown that the second order coherence properties of a certain class of fields may be computed within the framework of GO. The approach presented here may be expected to work well in cases where GO provides an adequate framework for computing the field propagated through an optical system. This result makes available for coherence theory a wealth of computational tools developed for applications in optical engineering. It should be appreciated that these methods may be applied in many disciplines in physics and engineering which are amenable to asymptotic analysis of wave fields. For instance, these results may be applied to problems in quantum mechanics where the system is described by mixed states where the classical trajectory, i.e., the geometrical approximation, dominates. The next order of approximation may be obtained by including secondary rays as in the geometrical theory of diffraction, or in the language of quantum mechanics by including second order fluctuations about the classical path.

For the computer code used to generate the results in this Letter, see Ref. [18].

This material is based upon work supported by the National Science Foundation under CAREER Grant No. 0239265. A. Zysk is supported by the Beckman Institute for Advanced Science and Technology. J. C. Schotland acknowledges support from the NSF under Grant No. DMR 0425780.

[1] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, England, 1995).
[2] L. S. Dolin, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **7**, 559 (1964) [Sov. Radiophys. **7**, 559 (1964)].
[3] J. Turunen and A. T. Friberg, Opt. Laser Technol. **18**, 259 (1986).
[4] S. A. Boppart, in *Optical Imaging and Microscopy: Techniques and Advanced Systems*, edited by P. Torok and F.-J. Kao (Springer-Verlag, Berlin, 2003), p. 309.
[5] T. E. Gureyev, D. M. Paganin, A. W. Stevenson, S. C. Mayo, and S. W. Wilkins, Phys. Rev. Lett. **93**, 068103 (2004).
[6] M. Born and E. Wolf, *Principles of Optics* (Cambridge University Press, Cambridge, England, 1999), 7th ed.
[7] M. Kline and I. W. Kay, *Electromagnetic Theory and Geometrical Optics* (John Wiley & Sons, New York, 1965).
[8] B. E. A. Saleh and M. C. Teich, *Fundamentals of Photonics* (John Wiley & Sons., New York, 1991).
[9] Y. Zhang, M. Sato, and N. Tanno, Opt. Lett. **26**, 205 (2001).
[10] W. H. Carter and E. Wolf, J. Opt. Soc. Am. **67**, 785 (1977).
[11] A. Starikov and E. Wolf, J. Opt. Soc. Am. **72**, 923 (1982).
[12] F. Gori, Opt. Comm. **34**, 301 (1980).
[13] H. T. Yura and S. G. Hanson, J. Opt. Soc. Am. A **4**, 1931 (1987).
[14] A. J. Devaney, A. T. Friberg, A. T. N. Kumar, and E. Wolf, Opt. Lett. **22**, 1672 (1997).
[15] D. G. Fischer and T. D. Visser, J. Opt. Soc. Am. A **21**, 2097 (2004).
[16] A. T. Friberg, T. D. Visser, W. Wang, and E. Wolf, Opt. Commun. **196**, 1 (2001).
[17] B. Lü, B. Zhang, and B. Cai, J. Mod. Opt. **42**, 289 (1995).
[18] http://optics.beckman.uiuc.edu